Stochastic Model Predictive Control for Building HVAC Systems: Complexity and Conservatism

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Abstract

This paper presents a stochastic model predictive control (SMPC) approach to building heating, ventilation, and air conditioning (HVAC) systems. The building HVAC system is modeled as a network of thermal zones controlled by a central air handling unit and local variable air volume boxes. In the first part of the paper, simplified nonlinear models are presented and validated for thermal zones and HVAC system components. The thermal load in each thermal zone includes occupancy and weather conditions. The uncertain load forecasts are modeled by finitely-supported probability density functions. The probability density functions of thermal loads are initialized by using historical data and updated as new data becomes available.

In the second part of the paper, we present a SMPC design that minimizes expected energy consumption and bounds the probability of thermal comfort violations. The SMPC uses predictive knowledge of the stochastic load in each zone during the design stage. The complexity of a commercial building requires special handling of system nonlinearities and chance constraints in order to enable real-time implementation, minimize energy consumption, and guarantee thermal comfort. The paper focuses on the trade-off between computational tractability and conservatism of the resulting SMPC scheme. The proposed SMPC scheme is compared with alternative SMPC designs, and the effectiveness of the proposed approach is demonstrated by simulation and experimental tests.

Index Terms

Stochastic model predictive control; Building energy system; Nonlinear system.

I. INTRODUCTION

In the past years there has been a renewed interest in modeling and predictive control for energy conversion, storage, and distribution in commercial buildings [1], [4], [15]–[19], [22]–[24], [29]–[35]. The majority of existing model predictive control (MPC) schemes has the objective of minimizing energy consumption while satisfying occupant thermal comfort using predictive knowledge of weather and occupancy (the building “load”). Simulative and experimental results have shown that closed-loop energy savings and comfort depend on the load disturbance predictions that are often different from actual realizations [15], [32].

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In this article, we discuss a stochastic model predictive control (SMPC) design to the aforementioned problem. In particular, we make use of stochastic information of weather and load learned from historical data, and minimize average energy consumption while bounding the probability of comfort violations. For commercial buildings of medium size, the simplest models reach the complexity of hundreds of states and control inputs. The complexity of the resulting SMPC problem motivates our research.

The following finite time stochastic optimization problem will be used to present our work.

\[
\begin{align*}
\min_{\kappa_t, \kappa_{t+1}, \ldots, \kappa_{T-1}} & \quad \sum_{k=0}^{k=T-1} \text{Energy}(\mathbb{E}(x_{t+k|t}), \mathbb{E}(u_{t+k|t}), \mathbb{E}(d_{t+k|t})) \\
\text{subj. to} & \quad x_{t+k+1|t} = f(x_{t+k|t}, u_{t+k|t}, d_{t+k|t}), \forall k \in \mathbb{N}_{T-1}, \forall d_{t+k|t} \in \mathcal{D} \\
& \quad u_{t+k|t} = \kappa_{t+k|t}(x_{t+k|t}), \forall k \in \mathbb{N}_{T-1}, \\
& \quad u_{t+k|t} \in \mathcal{U}, \forall k \in \mathbb{N}_{T-1}, \\
& \quad \mathbb{P}(x_{t+k|t} \in \mathcal{X}) \geq 1 - \epsilon, \forall k \in \mathbb{N}_T, \\
& \quad x_{t|t} = x(t),
\end{align*}
\]

where the symbol \( u_{t+k|t} \) is the random variable \( v \) at time \( t+k \) predicted at time \( t \), \( \mathbb{E}(v) \) denotes the expected value of \( v \), \( \mathbb{P}(v \in \mathcal{V}) \) is the probability of the event \( v \in \mathcal{V}, \mathbb{N}_T \) is the set of integers \( \{0, 1, \ldots, T\} \), \( x \) is the system state, \( u \) is the control input, and \( d \) is the uncertain load. The functions \( \text{Energy}(x, u, d) \) and \( f(x, u, d) \) define the energy consumption and system dynamics, respectively. \( \mathcal{D} \) represents the set of all possible realization of uncertain load \( d \). Problem (1) seeks a set of control feedback laws \( \kappa_{t+k|t} \in \mathcal{U} \) which minimizes the expected energy consumption (1a), and that has small probability of state constraint violation \( x \notin \mathcal{X} \) (1e).

The solution to the stochastic MPC problem (1) requires four main steps: (i) the translation of the optimization over control polices \( \kappa(\cdot) \) into a finite-dimensional optimization problem, (ii) the propagation of the stochastic system states over the prediction horizon, (iii) the translation of probabilistic constraints into deterministic constraints, and (iv) the solution of the resulting mathematical program.

With the exception of linear systems and special classes of distributions (e.g., normal distributions), steps (i)-(iii) are non-trivial and affect the complexity of the mathematical problem at step (iv) and the conservatism of the resulting control policy [8], [20], [28], [37], [42]. Step (i) has different solutions [3]. Open-loop prediction schemes are conservative since they look for one optimal open-loop control sequence that has to cope with all possible future disturbance realization, without taking future measurements into account. Closed-loop formulations overcome this issue but they can quickly lead to intractable problems. A compromise consists in fixing the control structure (e.g. affine state-feedback policies or affine disturbance feedback), parameterizing the control sequence in the feedback gains, and optimizing over these parameters [26], [27].

The chance constraints (1e) in step (iii) are translated into deterministic ones by enforcing tightened constraints on expected values of states and inputs. The tightening offset is computed based on the tails of the disturbance probability distributions [40], [41]. In practice, the distribution of the ambient temperature and occupancy load in
buildings are **upper and lower bounded by a finite number**, and it is non-Gaussian. In addition, the simplified control oriented models for buildings are bilinear systems [14], [21], [32], [35]. The bilinear terms arise from the multiplication of supply air mass flow rate and temperature to compute the cooling and heating energy delivered to thermal zones. Exact solutions to stochastic MPC problem (1) for nonlinear systems subject to non-Gaussian disturbances are, in general, computationally intractable for real-time implementation in HVAC systems. Obtaining a computationally tractable approximation to this problem is crucial for the real-time implementation of stochastic MPC. The authors in [35] proposed to compute the nonlinear SMPC problem by solving a sequence of tractable subproblems. For each subproblem, the nonlinear dynamics are linearized around the simulated trajectory, and the tightening offsets for chance constraints are computed by bounding the tails of Gaussian distributions. The resulting problems become quadratic programs if the linear disturbance feedback gain is optimized, and the problems become second order cone programs if the linear disturbance feedback gain is determined off-line.

Non-Gaussian and finitely-supported disturbances have been studied in [6], [7], [10], [11], [26], [27] in the context of SMPC. The authors in [10], [11] use a tube-based method to translate the chance constraints. The work in [26], [27] suggests to translate the chance constraints to deterministic ones by using tightening offsets. The offsets are computed off-line using numerical approximations of convolution integrals. Sample-based methods provide an alternative approach to transform the chance constraints for non-Gaussian random variables [5]–[7]. The approach consists in transforming the chance constraints (1e) into deterministic counterparts by evaluating them at a large number of disturbance samples.

In this study, we build on the work in [7], [26], [27] to design SMPC algorithms for networks of bilinear systems subject to non-Gaussian disturbances while retaining computational tractability. In particular, the bilinear system (1b) is linearized by using feedback linearization. The chance constraints (1e) then are transformed to deterministic ones by using two techniques: discrete convolution integrals and sample-based method. The complexity of both approaches is studied as a function of the problem size. The resulting nonlinear program is solved using Ipopt, a software package for large-scale nonlinear optimization problems [44].

The state feedback linearization proposed in this paper has several advantages. In particular, it allows us to easily optimize over a feedback policy. Also, the transformation of the chance constraints into deterministic ones does not depend on the linearized systems dynamics as in [35]. The proposed SMPC design is carefully analyzed and compared with existing approaches. In particular, in Section IV we discuss the tradeoff between performance, conservatism, and complexity. We will also try to shed some lights on the following questions.

**Q1** Does one need a stochastic MPC formulation, or nominal MPC with expected forecasts provides “good” performance for HVAC systems?

**Q2** Is there value in using nonlinear probability distribution functions or does Gaussian approximation work well, and what is the price one has to pay for it?

**Q3** Can the proposed approach be implemented in large scale buildings?

**Q4** Should one transform the chance constraints by using convolution integrals or sample-based methods? The effectiveness of the proposed approach will be demonstrated by using simulation and experiments. In both
simulative and experimental studies, models and probability distribution functions of prediction uncertainties are generated by using measured historical data.

II. SYSTEM MODEL

A. HVAC System

Figure 1 depicts the main equipment used to produce and distribute cold or hot air in a building. The air handling unit (AHU) recirculates return air from building spaces, and mixes it with fresh outside air. The proportion of return air to outside air is controlled by damper positions in the AHU. The mixed air is cooled by the cooling coils that extract the cooling energy from chilled water produced by chillers. The air temperature after the cooling coil depends

![Schematics of the air distribution system.](image)

on the cooling coil valve position, the temperature of the chilled water, the temperature of mixed air entering the cooling coil, the mass flow rate of the mixed air, and the physical characteristics as well as thermal effectiveness of the cooling coil. Cool air is delivered to building spaces by electrical fans. Before reaching a given space, the air goes through variable air volume (VAV) boxes. At each VAV box the mass flow rate of the air supplied to the space is adjusted by a damper position. In addition, air temperature can be increased using reheat coils installed in the VAV box when needed. Space served by one VAV box is denoted by a thermal zone. The delivered air enters a zone through diffusers that are designed to fully mix the incoming air with the air in the thermal zone.

In the next subsections we present our approach to AHU and VAV boxes modeling. A combination of reduced-order, physic-based models and static performance maps is employed. Their parameters are learned in real-time from measured data. Historical data allows us to quantify load and model uncertainty. The approach has appeared already in [32]. We included the main relevant ideas in this paper to improve its readability and to facilitate the understanding of the control design approach presented later in this paper.
B. Thermal Zone Model

An undirected graph structure is adopted to represent the thermal zones and their dynamic couplings in the following way. The $i$-th zone is associated with the $i$-th node of a graph, and when an edge $(i, j)$ connecting the $i$-th and $j$-th node is present, the thermal zones $i$ and $j$ are subject to direct heat transfer. The graph $G$ is defined as

$$G = (V, A),$$

(2)

where $V = \{1, \ldots, N_v\}$ is the set of nodes or vertices, and $A \subseteq V \times V$ is the set of edges $(i, j)$ with $i \in V$, $j \in V$. $N^i$ is the set of neighboring nodes of $i$. We model the temperature dynamics of each thermal zone $i \in V$ as an autoregressive-moving-average model with exogenous inputs model (ARMAX model) [21], [43].

$$T^{i}_{t+1} = \sum_{q=0}^{q_x} p_{1,q} T^{i}_{oa,t-q} + p_{2,q} I^{i}_{t-q} + p_{3,1} (T^{i}_{s,t} - T^{i}_t) \dot{m}_{s,t}^{i} + \sum_{q=0}^{q_d} \left( p_{4,q} T^{i}_{t-q} + \sum_{j \in N^i} p_{5,q,j} T^{j}_{t-q} \right) + p_{6} + P_{d,k}^{i}, \forall i \in V,$$

(3)

where $T^{i}_k$ is the temperature of zone $i$ at time $k$. For the zone $i$ at time $k$, $\dot{m}_{s,k}^{i}$ is the supply air mass flow rate, $T^{i}_{s,k}$ is the supply air temperature, $P_{d,k}^{i}$ is the internal thermal load, $T^{i}_{oa,k}$ is the ambient temperature, $I^{i}_k$ is the solar radiation intensity. The supply air mass flow rates and temperatures $\dot{m}_{s}^{i}$ and $T^{i}_s$ are control inputs. In (3) $T^{i}$ are system states, and $P_{d}^{i}$, $T^{i}_{oa}$ and $I^{i}$ are exogenous signals. In (3) $q_x$ and $q_d$ are the autoregressive order and moving average order, respectively. The selection of $q_x$ and $q_d$ affects the complexity and the accuracy of the model. The bilinear term $(T^{i}_{s}(t) - T^{i}(t))\dot{m}_{s}(t)$ in (3) is introduced to match first-order energy-balance equations. The bilinear regression model (3) is identified using data collected from unoccupied hours when the internal load $P_{d}$ is minimal.

In particular, the historical measure of zone temperatures, ambient temperature, solar radiation intensity, and recorded control inputs are used to propagate model (3) given the initial zone temperature, and the parameters $p$ are optimized by solving a linear regression problem so that the errors between the measured zone temperature and predicted zone temperature are minimized.

We have used model (3) with $q_x = q_d = 2$ to model the second floor of the Bancroft library at UC Berkeley. The values of $q_x$ and $q_d$ are selected so that the resulting model (3) enjoys the least identification error. The floor plan of the Bancroft library is depicted in Figure 2. Results for the classroom (labeled VAV C-2-15 in Figure 2) are reported next.

The identification results are reported in Figure 3 for the weekend of January 20, 2012, and the classroom is unoccupied during the weekends, i.e. $P_{d}^{i} \in V = 0$. The identified model then is validated for the weekend of January 21, 2012, and the results are plotted in Figure 4.

Next we provide more details on the generation of forecasts for thermal load $P_{d}^{i}$, ambient temperature $T^{i}_{oa}$, and solar radiation intensity $I^{i}$ in (3).
Fig. 2. Bancroft library floor plan

Fig. 3. Identification results of the thermal zone model (3)  
Fig. 4. Validation results of simplified room model (3)

C. Load Models

The internal thermal loads are calculated as follows. The zone model (3) is first identified using data from unoccupied hours. By comparing occupied hour data to the model predictions, we can capture thermal load by occupants. We define a load profile as the difference in temperature between a local model prediction and the measurements:

\[ P_{d,t-1}^i = (T_{\text{meas},t}^i - T_t^i), \quad \forall i \in \mathcal{V}, \]  

\( (4) \)
where \( P_{d,i}^{t} \) is the load for zone \( i \), \( T_{\text{meas},i}^{t} \) is the measured temperature of zone \( i \) at time \( t \), and \( T_{i}^{t} \) is the zone \( i \) temperature prediction with the identified model (3) using data at time-step \( t - 1 \).

**Remark 1:** The load estimated by (4) includes model errors. Alternative methods to estimate the loads for building spaces have been reported in literatures [2], [17], [36], and they can be easily incorporated into our framework with minor modifications.

To illustrate the effectiveness of the approach, next we report the results for a classroom of the Bancroft library at UC Berkeley (labeled as VAV C-2-15 in Figure 2). Figure 5 reports the occupancy load from May 2011 to February 2012 computed by (4). In Figure 5, each dot represents the estimated load at an individual hour within one week, and the solid line is the weekly mean occupancy load over the whole period (May 2011 to February 2012).

![Fig. 5. Occupancy load](image)

In Figure 5 we observe that during weekdays the average occupancy load profile has a peak load of 0.15°C, and there is no significant occupancy load during weekends. This phenomenon matches the occupancy schedule of the classroom. The model (4) is used to generate uncertainty map for an occupancy load predictor. The occupancy loads are predicted using a lookup table with period of one week,

\[
P_{d,k+t|t}^{i} = \hat{P}_{d,k+t|t}^{i} + \tilde{P}_{d,k+t|t}^{i}, \forall i \in \mathcal{V},
\]

where \( P_{d,k+t|t}^{i} \) is the predicted occupancy load at time \( t + k \) when the prediction starts from \( t \), \( \hat{P}_{d,k+t|t}^{i} \) is the mean of the internal load prediction at time \( t + k \) plotted as the solid line in Figure 5, and \( \tilde{P}_{d,k+t|t}^{i} \) is the prediction uncertainty described by a probability density function (PDF) with finite support. The finitely-supported probability density function can be computed from samples of \( P_{d}^{i} \) estimated through (4) by using simple piecewise constant approximation or kernel density estimation [39]. In particular, the load samples in Figure 5 will have a bimodal distribution at time instances between Thursdays and Fridays due to occasionally conferences in the afternoons.
The solar load for each zone is computed as the sun radiation intensity projected onto the normal vector to the outside wall. Figure 6 reports the computed daily sun load for an office room (labeled as VAV C-2-5 in Figure 2) from December 01, 2011 to January 31, 2012 in the DOE library at UC Berkeley. In Figure 6, the dots are the computed daily sun load samples, and the solid line denotes the mean value of the computed daily solar load. We use a periodic solar load $I_{i,k+t|t}$ in zone $i$ prediction model with period of one day. The mean $\hat{I}_{i,k+t|t}$ and the probability density function of the prediction uncertainty $\tilde{I}_{i,k+t|t}$ are calculated from historical data.

The ambient temperature at time $t + k$ predicted at $t$ is

$$T_{oa,k+t|t} = \hat{T}_{oa,k+t|t} + \tilde{T}_{oa,k+t|t}$$

where $\hat{T}_{oa,k+t|t}$ is weather forecast obtained from the local weather station, and $\tilde{T}_{oa,k+t|t}$ is weather forecast uncertainty. The weather forecast error is defined as the difference between historical weather measurements and archived weather forecast data, and the probability density function of the ambient temperature prediction uncertainty $\tilde{T}_{oa}$ is modeled from historical weather forecast errors. In our study, the prediction uncertainty $\tilde{T}_{oa}$ is learned from archived weather prediction and measurements from May 20, 2012 to July 07, 2012. Figure 7 shows the probability density functions of the weather forecast uncertainty for four selected prediction times $k$. As expected, the measured ambient temperature forecast uncertainty increases with longer prediction horizon $k$.

Remark 2: The approach presented in this section to forecast the building loads and to estimate their uncertainty is simple. The control design methodology presented in the following section does not depend on the specifics of the forecast algorithms. It only requires probability distribution functions of forecasted loads. Different methods could be used to obtain more accurate predictions.
Fig. 7. Ambient temperature prediction error

D. Energy Model

The AHU and VAV components that use energy are dampers, supply fans, and heating coils. The supply fan needs electrical power to drive the system, the heating coils consume the energy of hot water, and the power to drive the dampers is assumed to be negligible. The fan power can be approximated as a second order polynomial function of the total supply air mass flow rate ($\dot{m} = \sum_{i \in \mathcal{V}} \dot{m}_s^i$) driven by the fan.

$$P_{\text{fan}} = c_0 + c_1 \dot{m} + c_2 \dot{m}^2,$$

where $c_0$, $c_1$, $c_2$ are parameters to be identified by fitting recorded data. Heating and cooling coils are air-water heat exchangers, and the power consumption of the coils are derived from the energy conservation law,

$$P_h = p_h \sum_{i \in \mathcal{V}} \dot{m}_s^i c_p (T_s^i - T_c), \quad P_c = p_c \sum_{i \in \mathcal{V}} \dot{m}_s^i c_p (T_m - T_c)$$

where $P_c$ ($P_h$) is the power used by the cooling (heating) coils to deliver supply air with temperature $T_s^i$, $p_c$ and $p_h$ are model parameters, $T_c$ is the air temperature after the AHU cooling coil, and $T_m$ is the mixed air temperature before the cooling coil (see Figure 1). The mixed air temperature is computed as

$$T_m = \delta T_{\text{oa}} + (1 - \delta) \frac{\sum_{i \in \mathcal{V}} \dot{m}_s^i T^i}{\sum_{i \in \mathcal{V}} \dot{m}_s^i},$$

where $\delta$ is the mixing ratio between the outside air and return air. It is assumed that the return air temperature is a weighted sum of the zone temperatures with weights being the mass flow rates supplied to the corresponding
zones. Both $\delta$ and $T_c$ can be controlled through the AHU cooling coil and return damper. The total electricity power consumption of the HVAC system at time $t$ then is calculated as

$$P_{\text{tot}} = (P_h + P_c + P_{\text{fan}}).$$

(10)

E. Constraints

The HVAC system is subject to thermal comfort constraints and operational constraints defined next.

C1- $P \{ T_i^k \geq T_i^k \} > 1 - \epsilon$, $P \{ T_i^k \leq T_i^k \} > 1 - \epsilon$, $\forall i \in \mathcal{V}$. The probability that zone temperatures at time step $k$ are within the comfort bounds is greater than $1 - \epsilon$. Comfort bounds $T_i^k$, $T_i^k$ and allowed violation probability $\epsilon$ are design parameters.

C2- $T_{s,k} \in [T_{i}^s, T_{i}^s]$, $\forall i \in \mathcal{V}$. The supply air temperature is limited by the chilled water temperature through the coils and the physical characteristics of coils.

C3- $\dot{m}_{s,k}^i \in [\dot{m}_{s,k}^i, \dot{m}_{s,k}^i]$, $\forall i \in \mathcal{V}$. Allowed mass flow rate for the supplied air. The lower bounds $\dot{m}_{s,k}^i$ are strictly positive to meet minimum ventilation requirement.

C4- $\sum_{i \in \mathcal{V}} \dot{m}_{s,k}^i T_c \leq \sum_{i \in \mathcal{V}} \dot{m}_{s,k}^i T_m$. The air temperature after the cooling coil $T_c$ cannot be warmer than the mixed air temperature $T_m$ computed by (9).

C5- $\dot{m}_{s,k}^i T_c \leq \dot{m}_{s,k}^i T_s^i$, $\forall i \in \mathcal{V}$. The air temperatures across the heating coil can only increase.

F. Model Summary

The bilinear ARMAX model (3) for all zones and the energy model (10) can be compactly rewritten as

$$x_{k+1} = f(x_k, u_k, w_k), w_k \in \mathcal{W}_k,$$

(11a)

$$T_k = C x_k, \forall i \in \mathcal{V},$$

(11b)

$$P_{\text{tot},k} = P_{\text{tot}}(x_k, u_k, w_k),$$

(11c)

where $x_k = [T^1_k, T^{-1}_k, T^1_{k-1}, T^{-1}_{k-1}, \ldots, T^1_{N_c}, T^{-1}_{N_c}, T^1_{N_c}, T^{-1}_{N_c}] \in \mathbb{R}^{N_x \times 1}$ is the system state, $u_k = [\dot{m}_{s,k}^1, T^1_{s,k}, \ldots, \dot{m}_{s,k}^{N_c}, T_{s,k}^{N_c}, T_{c,k}, \delta_k] \in \mathbb{R}^{n_u \times 1}$ is the control input, and $w_k = [P_{d,k}^1, \ldots, P_{d,k}^{N_c}, I_{k-1}^1, I_{k-1}^1, \ldots, I_{k-1}^{N_c}, I_{k-1}^{N_c}, I_{oa,k}, I_{oa,k-1}, I_{oa,k-2}] \in \mathbb{R}^{n_d \times 1}$ is the system load. The forecasts of $w_k$ are obtained as discussed in Section II-C. The state vector $x_k$ includes the state of the ARMAX model (3) at every zone of the building. The input vector $u_k$ includes the supply air mass flow rate $\dot{m}_{s,k}$ and the supply air temperature $T_{s,k}$ to each zone, as well as the air temperature after the AHU cooling coil and the mixing ratio between the outside air and return air.

The constraints C1–C5 are compactly written as

$$P \{ h^T x_k \leq g_k^j \} > 1 - \epsilon, \forall j \in \mathbb{N}_{N_c},$$

(12a)

$$G(x_k, u_k, w_k) \leq 0, \forall w_k \in \mathcal{W}_k,$$

(12b)

where $N_c = 2N_v$ is the number of state inequality constraints, $\mathbb{N}_{N_c}$ is the index set $\{0, 1, \ldots, N_c\}$, and $v^T$ is the transpose of a vector $v$. 
III. Stochastic MPC

Consider the building model (11) and its constraints (12). We formulate the following stochastic optimization problem with chance constraints for comfort constraints:

$$\min_{X_t, U_t} \sum_{k \in \mathbb{N}_{T-1}} P_{tot}(\hat{x}_{k|t}, \hat{u}_{k|t}, \hat{w}_{k|t}) \Delta t,$$

subject to:

$$x_{k+1|t} = f(x_{k|t}, u_{k|t}, w_{k|t}), \forall k \in \mathbb{N}_T,$$

$$\mathbb{P}\{h^T \mathbf{p}_{k|t} \leq g_k\} \geq 1 - \epsilon, \forall j \in \mathbb{N}_N, \forall k \in \mathbb{N}_T,$$

$$G(x_{k|t}, u_{k|t}, w_{k|t}) \leq 0, \forall w_{k|t} \in W_{k|t}, \forall k \in \mathbb{N}_{T-1},$$

$$x_0 = x(t),$$

where $X_t = \{x_0|t, \ldots, x_T|t\}$ and $U_t = \{u_{0|t}, \ldots, u_{T-1|t}\}$ are the optimization variables and $\Delta t$ is the sampling time. We use the notation $v_{t+k|t}$ to denote the value of the variable $v$ at time $t+k$ predicted at time $t$. Similarly, $\hat{v}_{t+k|t}$ is the expected value of the random variable $v$ at time $t+k$ predicted at time $t$.

A stochastic model predictive controller solves problem (13) at each time step $t$. In particular, let $X^*_t = \{x^*_0|t, \ldots, x^*_T|t\}$ and $U^*_t = \{u^*_{0|t}, \ldots, u^*_{T-1|t}\}$ be the optimal solution of problem (13) at time $t$. Then, the first element of optimal control sequence is implemented to the system (11a),

$$u^*_{t|t} = u^*_{0|t}.$$

The optimization problem (13) is repeated at $t = t + \Delta t$, with the updated state $x_0 = x_{t+1}$.

The formulation (13) has the following features; (i) the power consumption at expected values of states, inputs and disturbances is minimized in (13a), (ii) chance constraints (13c) are used for temperature bounds, (iii) constraint C2-C5 are robustly enforced for all admissible disturbances realizations in (13d). As it will be clear from the results presented later in this paper, this formulation will enable complexity reduction (by avoiding the computation of nonlinear probability distribution functions) while reducing the conservatism typically associated with the use of simple linear models, gaussian uncertainty, and robust temperature bound satisfaction.

The next section presents the main contributions of this work.

1) Propose a computationally tractable approach to solve the stochastic MPC problem (13) while reducing conservatism. We first make use of a state feedback linearization to linearize the bilinear system model (13b). The chance constraints then can be reformulated as deterministic constraints by bounding the distribution tails. The resulting non-convex optimization problem is solved using Ipopt [44].

2) Compare the complexity and conservatism of two methods used to reformulate the chance constraints (13c) with deterministic constraints: discrete method and sample-based method.

3) Carry out extensive simulation tests to demonstrate the effectiveness of the proposed SMPC scheme compared with alternative MPC designs.

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Remark 3: With abuse of notation and for the sake of simplicity, in the rest of the paper, the notation $x_{k+l|t}$ will be replaced with $x_k$.

A. System Linearization

The bilinear dynamic model (13b) is linearized by introducing deterministic virtual inputs $u^i_s$ and $u^i_z$ for each zone $i$,

$$u^i_s = \dot{m}^i_{s,k} T^i_{s,k}, \forall i \in V,$$  \hspace{1cm} (14a)

$$u^i_z = \dot{m}^i_{s,k} T^i_{k}, \forall i \in V,$$  \hspace{1cm} (14b)

to obtain the linear model

$$x_{k+1} = Ax_k + Bu_k + Dw_k, \quad w_k \in W_k,$$  \hspace{1cm} (15)

$$T^i_k = C^i x_k, \forall i \in V,$$  \hspace{1cm} (16)

where the virtual control input vector $u_k$ is $u_k = [u^1_{s,k}, u^1_{z,k}, \ldots, u^{N_v}_{s,k}, u^{N_v}_{z,k}, T_{c,k}, \delta_k]$. For a given virtual input vector $u_k$, the input signals $u^{|nl}_k$ can be uniquely determined if $T^i \neq 0$, $\dot{m}^i_s \neq 0$, $\forall i \in V$. Any feasible system trajectory satisfies these two constraints (see Section II-E).

In model (15) $w_k$ is a random variable with bounded support $W_k$. Model (15) implies that the dynamics of state mean $\bar{x}$ and state error $\tilde{x} = x - \bar{x}$ are

$$\bar{x}_{k+1} = A \bar{x}_k + Bu_k + D\tilde{w}_k,$$  \hspace{1cm} (17a)

$$\bar{T}^i_k = C^i \bar{x}_k,$$  \hspace{1cm} (17b)

$$\bar{x}_{k+1} = A \bar{x}_k + D\tilde{w}_k,$$  \hspace{1cm} (17c)

$$\bar{T}^i_k = C^i \bar{x}_k,$$  \hspace{1cm} (17d)

$$\tilde{x}_0 = 0.$$  \hspace{1cm} (17e)

The linear system (17) can be used in the optimization problem (13) instead of the nonlinear dynamics (13b). The new control variables $u^i_{s,k}$ in (14) can be interpreted as a deterministic state-feedback gain linking mass air flow rate $\dot{m}^i_s$ and supply air temperature $T^i_s$ to room temperature $T^i$: 

$$\dot{m}^i_{s,k} = u^i_{s,k} \frac{1}{T^i_k}, \forall i \in V.$$  \hspace{1cm} (18a)

$$T^i_{s,k} = u^i_{s,k} \frac{1}{\dot{m}^i_{s,k}} = u^i_{s,k} T^i_{k}, \forall i \in V.$$  \hspace{1cm} (18b)

Therefore, while the optimization in the virtual inputs $u^i_s$ and $u^i_z$ is over open-loop policies [3], the variable substitution (14) provides state-feedback policies. These advantages come with a price. In fact, the energy models in Section II-C and constraints defined in Section II-E have to be rewritten as a function of the virtual inputs $u_k$. 

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By using the change of control variables (14) the energy models for supply fan (7), cooling and heating coils (8) are rewritten as

\[ P_{fan} = c_0 + c_1 \sum_{i \in V} \frac{u_i^i}{T_i} + c_2 \left( \sum_{i \in V} \frac{u_s^i}{T_i} \right)^2, \]  

\[ P_h = p_h c_p \sum_{i \in V} \left( u_s^i - \frac{u_s^i}{T_i} T_e \right), \]  

\[ P_c = p_c c_p \sum_{i \in V} \left( \frac{u_i^i}{T_i} \delta T_{oa} + (1 - \delta) u_s^i - \frac{u_s^i}{T_i} T_e \right), \]  

\[ P_{tot} = (P_{fan} + P_c + P_h) = P_{tot}(x, u, w). \]  

The constraints on supply air temperature \( C_2 \) are rewritten as the following bilinear constraints,

\[
\max_{\tilde{u}_z, k} u_{s, k}^i T_k^i = u_{s, k}^i C^i \tilde{x}_k + u_{s, k}^i \tilde{T}_k^i, \quad \forall i \in V
\]

\[
\min_{\tilde{u}_z, k} u_{s, k}^i T_k^i = u_{s, k}^i C^i \tilde{x}_k + u_{s, k}^i \tilde{T}_k^i, \quad \forall i \in V.
\]

The constraints on supply air mass flow rate \( C_3 \) are rewritten as the following robust constraints

\[
u_{s, k}^i \leq \min_{\tilde{u}_k \in \tilde{u}_{k-1}} \tilde{m}_{s, k} T_k^i = \tilde{m}_{s, k} C^i \tilde{x}_k + \tilde{m}_{s, k} \tilde{T}_k^i, \quad \forall i \in V,
\]

\[
u_{s, k}^i \geq \max_{\tilde{u}_k \in \tilde{u}_{k-1}} \tilde{m}_{s, k} T_k^i = \tilde{m}_{s, k} C^i \tilde{x}_k + \tilde{m}_{s, k} \tilde{T}_k^i, \quad \forall i \in V,
\]

where

\[
\tilde{T}_{k, min} = \min_{\tilde{u}_k \in \tilde{u}_{k-1}} C^i \left( \tilde{x}_0 + \sum_{i=0}^{k-1} A^k D \tilde{u}_{k-1-i} \right),
\]

\[
\tilde{T}_{k, max} = \max_{\tilde{u}_k \in \tilde{u}_{k-1}} C^i \left( \tilde{x}_0 + \sum_{i=0}^{k-1} A^k D \tilde{u}_{k-1-i} \right).
\]

The constraints \( C_4 \) on the air temperature after the cooling coil in AHU \( T_c \) can be conservatively approximated as follows

\[
\max_{\tilde{u}_z, k} \left( \sum_{i \in V} \frac{u_{s, k}^i}{T_k^i} (T_c, k - \delta_k T_{oa,k}) + (\delta_k - 1) u_{s, k}^i \right)
\]

\[
\leq \sum_{i \in V} \frac{u_{s, k}^i}{C^i \tilde{x}_k + \tilde{T}_k^i} T_c, k - \delta_k T_{oa,k} \frac{u_{s, k}^i}{C^i \tilde{x}_k + \tilde{T}_k^i} + (\delta_k - 1) u_{s, k}^i \]

\[
\leq 0,
\]

where \( T_{oa,k} \) is the lower bound of \( T_{oa,k} \). The constraints \( C_5 \) on the air temperature after the cooling coil in AHU \( T_c \) become

\[
u_{s, k}^i T_c \leq \min_{\tilde{u}_z, k} u_{s, k}^i T_c = u_{s, k}^i C^i \tilde{x}_k + u_{s, k}^i \tilde{T}_k^i, \quad \forall i \in V.
\]

The feedback linearization (14) allows the reformulation of robust nonlinear input constraints \( C_2 \)–\( C_5 \) to deterministic nonlinear constraints (23)–(30), and they can be compacted as follows.

\[
G^{FE}(\tilde{x}_k, u_k, \tilde{u}_k) \leq 0, \forall k \in \mathbb{N}_T.
\]

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B. Solving Chance Constraints

In this section, we show the approach we used to transform the chance constraints (13c) to convex deterministic ones with tightening offsets [12]. We denote by \( \pi^j_k \) the univariate random variable \( h^j T x_k \). The chance constraints (13c) then can be reformulated as deterministic ones [13]

\[
\tilde{\pi}^j_k \leq g^j_k - \alpha^j_k, \quad \forall j \in \mathbb{N}_{N_e}, \quad \forall k \in \mathbb{N}_T,
\]

where \( \tilde{\pi}^j_k \) is the expected value of \( \pi^j_k \), and the offset \( \alpha^j_k \) is calculated as follows

\[
\int_{\alpha^j_k}^{\infty} \text{pdf}_{\pi^j_k}(y) dy = 1 - cdf_{\pi^j_k}(\alpha^j_k) = \epsilon, \quad \forall j \in \mathbb{N}_{N_e}, \quad \forall k \in \mathbb{N}_T,
\]

where \( \text{pdf}_{\pi^j_k}(y) \) is the probability density function of \( \pi^j_k = \pi^j_k - \tilde{\pi}^j_k \) evaluated at \( y \), and \( cdf_{\pi^j_k}(\alpha^j_k) = \mathbb{P} \{ \pi^j_k \leq \alpha^j_k \} \) is the cumulative density function of \( \pi^j_k \) evaluated at \( \alpha^j_k \). A solution to (32) is also a solution to (13c).

The univariate random variable \( \tilde{\pi}^j_k \) can be expressed as a linear function of previous disturbances

\[
\tilde{\pi}^j_k = h^j T \tilde{x}_k = h^j T \bar{x}_0 + \sum_{i=0}^{k-1} h^j T A^k D \tilde{w}_{k-1-i}.
\]

If the measurement of the state \( x_0 \) is exact, i.e. \( \bar{x}_0 = 0 \), then equation (34) is compactly written as

\[
\tilde{\pi}^j_k = \phi_k^j T \bar{W}_{k-1}, \quad \forall j \in \mathbb{N}_{N_e}, \quad \forall k \in \mathbb{N}_T,
\]

where \( \bar{W}_{k-1} = [\tilde{w}_{0}^T, \tilde{w}_{1}^T; \ldots; \tilde{w}_{N_{w}-1}^T]^T \) is a column vector of length \( N_w = n_d k \). The disturbances error \( \bar{W}_{k-1} \) is finitely-supported with lower bound \( \underline{W}_{k-1} \) and upper bound \( \bar{W}_{k-1} \).

Next we present and compare two numerical methods to compute the offset \( \alpha^j_k \) computed by (33) given the probability density function for the disturbances error \( \tilde{w} \).

1) Discrete Convolution Integral Method: If the disturbance errors \( \tilde{w} \) are statistically independent, the probability density function \( \text{pdf}_{\tilde{\pi}^j_k} \) of \( \tilde{\pi}^j_k \) in (35) can be computed recursively as follows

\[
\text{pdf}_{\tilde{\pi}^j_k}^{(1)} = \text{pdf}_{w(1)}(\cdot),
\]

\[
\text{pdf}_{\tilde{\pi}^j_k}^{(n+1)} = \text{pdf}_{\tilde{\pi}^j_k}^{(n)} * \text{pdf}_{w(n+1)}(\cdot), \quad n = 1, 2, \ldots, N_{w} - 1,
\]

where \( \text{pdf}_{w(n)} \) is the probability density function for \( n \)th element of \( \text{diag}(\phi_k^j) \bar{W}_{k-1} \) in (35). The operator * denotes the convolution integral.

In general, there is no analytical solution for convolution integrals (37). Numerical algorithms to approximate the convolution integrals (37) can be found in [9], [26], [27]. We used the approach in [26]. The probability density function \( \text{pdf}_{w(n)} \) in (37) is first discretized over \( N_s \) evenly distributed samples from \( [\underline{\pi}, \bar{\pi}] \) with discretization interval \( \Delta \pi = \frac{\pi - \underline{\pi}}{N_s} \).
The upper bound $\pi$ and lower bound $\bar{\pi}$ of the discretization range are computed as

$$
\pi_j^T = \max_{W_{T-1} \leq \bar{W}_{T-1} \leq W_{T-1}} \left( \phi_j^T \bar{W}_{T-1} \right), \forall j \in \mathbb{N}_e, \tag{38a}
$$

$$
\bar{\pi}_j^T = \min_{W_{T-1} \leq \bar{W}_{T-1} \leq W_{T-1}} \left( \phi_j^T W_{T-1} \right), \forall j \in \mathbb{N}_e, \tag{38b}
$$

$$
\bar{\pi} = \max_{j \in \mathbb{N}_e} \bar{\pi}_j^T, \quad \pi = \min_{j \in \mathbb{N}_e} \pi_j^T. \tag{38c}
$$

Within the discretization interval, the probability density function $pdf_{w(n)}$ is kept constant

$$
pdf_{w(n)}[k] = pdf_{w(n)}(\pi_k^T + k \Delta \pi), \forall k \in \mathbb{N}_N, \forall n \in \mathbb{N}_w. \tag{39}
$$

The convolution integral (37) is then approximated by discrete convolution,

$$
(pdf_{\pi_k} * pdf_{w(n+1)})[k] = \sum_{i \in \mathbb{N}_N} pdf_{\pi_k}^i[l] pdf_{w(n+1)}[k - l] \Delta \pi, \forall k \in \mathbb{N}_N. \tag{40}
$$

The probability density function $pdf_{\pi_k}$ in (37) is then obtained by approximating each convolution integral in (37) using discrete convolution (40). Once $pdf_{\pi_k}$ in (37) is computed, the tightening offsets $\alpha_k^j$ in (33) are computed as

$$
\alpha_k^j = \pi_k^j + \bar{\pi}_k \Delta \pi, \quad \bar{\pi}_k = \min \left\{ \bar{\pi}_k \right\} \geq 1 - \epsilon \right\}. \tag{41}
$$

The discrete convolution integral method to compute the tightening offsets $\alpha_k^j$ in (32) is summarized as follows.

Algorithm 1 (Discrete method):

S1 Compute the upper and lower bounds of the discretization range as in (38), and select the sample number $N_s$.

S2 Discretize the probability density function $w(n)$, $\forall n \in \mathbb{N}_w$ as in (39).

S3 Approximate the probability density function $pdf_{\pi_k}$ by discrete convolution as in (40).

S4 Compute the tightening offsets $\alpha_k^j$ as in (41).

2) Sample-based Method: The sampling-based approach to single-stage chance constrained problems (CCP) has been studied in [6], [5], [7]. Recently it has been extended to multiple-stage chance constrained problems [38]. It provides an alternative solution to the computation of the offset $\alpha_k^j$ in (33). The sampling-based approach transforms the chance constraints (13c) into a deterministic ones by generating for each individual constraint a large number $N_{s,k}^j$ of independent identically distributed disturbance samples $\tilde{w}_{k,1}^{j,1}, \tilde{w}_{k,2}^{j,2}, ..., \tilde{w}_{k,N_{s,k}^j}^{j}$ (usually referred to as scenarios) according to the probability density function of the disturbance. We use (34) and replace the chance constraints (33) with

$$
h_j^T \hat{x} + \sum_{i=0}^{k-1} h_j^T A^k D \tilde{w}_{k-1-i}^{j,l} \leq g_k^j, \forall j \in \mathbb{N}_N, \forall k \in \mathbb{N}_T, \forall l \in \mathbb{N}_{N_{s,k}^j}. \tag{42}
$$

For a sufficiently large $N_{s,k}^j$, the satisfaction of (42) guarantees that each chance constraint in (13c) is satisfied with probability of $1 - \beta_k^j$.

Such an approach introduces significant conservatism, since a solution satisfying the constraints for “many” disturbance realizations is close to a worst-case solution (i.e. $\epsilon$ in (13c) is close to zero). In order to reduce the
conservatism of the solution, a certain number of generated samples $N_{r,k}^j$ have to be removed from the set of samples without a significant loss of reliability $(1-\beta_k^j)$ of the solution. The link between the number of original samples $N_{s,k}^j$, the removed ones $N_{r,k}^j$, the allowable violation probability $\epsilon$ and the reliability parameter $\beta_k^j$ has been studied in [6], [5], [7], [38]. Here we use the inequality [38]:

$$\left(\frac{N_{r,k}^j + \zeta_k^j}{N_{r,k}^j}\right) \sum_{l=0}^{N_{s,k}^j - 1} \left(\frac{N_{s,k}^j}{l}\right) \epsilon^l (1-\epsilon)^{N_{s,k}^j-l} \leq \beta_k^j,$$

(43)

where $\zeta_k^j$ is the support rank of each individual constraint in (13c). The support rank of a chance constraint is the dimension of the vector space spanned by the constraint. In summary the chance constraint (13c) is transformed into

$$h^T \hat{x}_k \leq g_k^j - \alpha_k^{j,l}, \forall l \in \mathbb{I}^{N_{s,k}^j} \setminus \mathbb{I}^{N_{r,k}^j}, \forall k \in \mathbb{N}_T, \text{ and } \forall j \in \mathbb{N}_{N_c},$$

(44)

with offsets $\alpha_k^{j,l}$ calculated as:

$$\alpha_k^{j,l} = \sum_{i=0}^{k-1} h^T A_{i,l} D \hat{w}^{k-1-i},$$

(45)

where $\mathbb{I}^{N_{s,k}^j} = \{1, ..., N_{s,k}^j\}$ and $\mathbb{I}^{N_{r,k}^j} \subseteq \mathbb{I}^{N_{s,k}^j}$ is a set containing the indices of $N_{r,k}^j$ removed samples.

Since the support rank of the constraints (44) is $\zeta_k^j = 1$, inequality (43) reduces to:

$$\sum_{l=0}^{N_{s,k}^j - 1} \left(\frac{N_{s,k}^j}{l}\right) \epsilon^l (1-\epsilon)^{N_{s,k}^j-l} \leq \beta_k^j.$$

(46)

In order to ensure that the solution to the overall optimization problem is a feasible solution to the original optimization problem with high probability $1-\beta$, the $\beta_k^j$ for the $j$-th constraint and the $k$ time index have to will be selected to satisfy

$$\sum_{j \in \mathbb{N}_{N_c}} \sum_{k \in \mathbb{N}_T} \beta_k^j < \beta.$$

(47)

From equation (44), one can easily notice that the optimal sample removal strategy corresponds to the removal of the $N_{r,k}^j$ largest offsets $\alpha_k^{j,l}$ (notice that for fixed $j \in \mathbb{N}_{N_c}$ and $k \in \mathbb{N}_T$ all constraints in (44) are parallel), resulting in the least conservative approximation of the corresponding individual chance constraint in (13c). Finally, the offsets $\alpha_k^j$ in (33) are calculated as:

$$\forall k \in \mathbb{N}_T, \text{ and } \forall j \in \mathbb{N}_{N_c}, \alpha_k^j = \max(\alpha_k^{j,l}), l \in \mathbb{I}^{N_{s,k}^j} \setminus \mathbb{I}^{N_{r,k}^j}$$

(48)

The algorithm to compute $\mathbb{I}^{N_{s,k}^j} \setminus \mathbb{I}^{N_{r,k}^j}$ in (44) and the corresponding offsets $\alpha_k^j$ is summarized next.

Algorithm 2 (Sampling-based method):

S1 $\forall k \in \mathbb{N}_T$ and $\forall j \in \mathbb{N}_{N_c}$ compute the number of samples to generate $N_{s,k}^j$ and to remove $N_{r,k}^j$ that satisfy inequalities (46) and (47).

S2 $\forall k \in \mathbb{N}_T, \forall j \in \mathbb{N}_{N_c}$ and $\forall l \in N_{s,k}^j$ do

S2.1 Generate the set of samples $\mathbb{W}_k^j = \{\hat{w}_k^{j,l}\}$,

S2.2 Calculate the set of the offsets $\{\alpha_k^{j,l}\}$ using (45),
S2.3 Sort the set of the offsets to obtain ordered set of the offsets \( \{ \bar{\alpha}^{j,l}_{k} \} \)

S3 \( \forall k \in \mathbb{N}_T \) and \( \forall j \in \mathbb{N}_{N_c} \) do

S3.1 Remove \( N_{r,k}^{j} \) largest offsets from the set \( \{ \bar{\alpha}^{j,l}_{k} \} \),

S3.2 Calculate the offsets \( \alpha^{j}_{k} \) using (48).

C. Optimization Problem

With the change of variables presented in Section III-A, and the transformation of chance constraints in Section III-B, the stochastic MPC problem (13) is transformed into the deterministic nonlinear optimization problem

\[
\min_{\hat{x}_{k+1},u_{k},\hat{w}_{k}} \sum_{k \in \mathbb{N}_{T-1}} P_{\text{tot}}(\hat{x}_k, u_k, \hat{w}_k) \Delta t, \tag{49a}
\]

subj. to:

\[
\hat{x}_{k+1} = A\hat{x}_k + Bu_k + D\hat{w}_k, \forall k \in \mathbb{N}_T, \tag{49b}
\]

\[
h^{T} \hat{x}_k \leq g_k - \alpha^{j}_{k}, \forall j \in \mathbb{N}_{N_c}, \forall k \in \mathbb{N}_T, \tag{49c}
\]

\[G^{FL}(\hat{x}_k, u_k, \hat{w}_k) \leq 0, \forall k \in \mathbb{N}_{T-1}, \tag{49d}\]

\[
\hat{x}_0 = x(0). \tag{49e}
\]

We compute the optimal solution \( u^{*}_{k \in \mathbb{N}_T} \) to Problem (49) by using Ipopt [44]. We remark that the steps presented in the previous section are crucial for obtaining a resulting problem (49) which is computationally tractable and whose solution is not too conservative. In the next section we will provide metrics from conservatism and computational tractability. We will also present results which confirm the effectiveness of our approach.

IV. Results

The stochastic MPC algorithm solving (49) at every time step \( k \) is analyzed in this section. For the class of building HVAC systems studied in this paper, we will investigate the questions Q1–Q4 raised in Section I. Although the answers depend on the HVAC system and the level of uncertainty, the next sections will shed some light on the aforementioned questions by simulation and experiments. In both simulation and experiments study, models and probability distribution functions of forecast uncertainty are generated by using measured historical data.

A. Bancroft Library Simulations

1) Simulation Setup: We present closed loop simulation of several MPC algorithms for the second floor of the Bancroft library at UC Berkeley. The second floor plan of the Bancroft library is depicted in Figure 2 and consists of classrooms, offices, and conference rooms. The HVAC system has a single duct configuration shown in Figure 1, and it is controlled and monitored using WebCTRL® developed by Automated Logic Corporation.
The original stochastic nonlinear control problem (13) is not feasible for real-time implementation for this system using current desktop capabilities. The goal of this section is to study the compromise between complexity, performance, and conservatism of different approximations to (13), including the one proposed in this paper.

We call the control logic implemented by WebCTRL® the baseline control (BC). The BC has been fine-tuned by professional and implements a “trim and respond” control algorithm. Details on the BC algorithm can be found in [25]. The main idea is to control the heating coil valve position command and the airflow set point as a function of the difference between zone temperature and the bounds on comfort level. When the zone temperature is within the comfort range, the VAV box maintains the minimum ventilation level. If a zone violates the thermal comfort upper-bound, a cooling request is triggered. If a zone violates the thermal comfort lower-bound, a heating request is triggered. The total number of cooling requests and heating requests is then used to control the AHU unit.

We compare the measured BC performance with the following MPC algorithms.

L1 Perfect MPC control (PMPC). A model predictive controller with perfect knowledge of disturbance prediction, i.e. the disturbance prediction error \( \tilde{w}_t = 0 \), and the mean value of the predicted disturbance over the horizon \( \hat{w}_t \) is equal to the future disturbance realization. This is not physically implementable but will be used as reference for what could be potentially achieved without uncertainty.

L2 Certainty equivalent MPC (CMPC). A nominal model predictive controller that solves the optimization problem (49) with the chance constraint offsets \( \alpha^j_k = 0 \), and the predicted disturbance \( \hat{w}_t \) set to be the mean value of load profiles modeled in Section II-C, i.e. the solid line in Figure 5. Clearly, in closed-loop simulations the nominal disturbance prediction might be different from the actual disturbance realization.

L3 Stochastic MPC (ESMPC) solving (49) with chance constraints approximated using the discrete convolution method in Section III-B1.

L4 Stochastic MPC (SSMPC) solving (49) with chance constraints approximated using the sample-based method in Section III-B2.

L5 Stochastic MPC (GSMPC) solving (49) with disturbances probability density function approximated as Gaussian distributions.

We focus on eight zones in the second floor of Bancroft library from December 1, 2011 to February 1, 2012. The eight zones are subject to negligible thermal interaction between each other. The zone model parameters are identified by linear regression using historical weekend data from December 2011 to February 2012.

In our simulations, the control parameters for the SMPC controllers (13) are set as follow. The control sampling time \( \Delta t \) is 15 minutes, the comfort constraint is allowed to violate with a chance of \( \varepsilon = 5\% \), the prediction horizon is \( T = 20 \) (5 hours).

The thermal comfort constraints (\( T^u_t \) and \( T^l_t \)) are zone dependent and have a period of one day. Rather than using the original system comfort constraints, we will use what the BC actually achieved. This approach allows us to properly compare the MPC and BC control performances. In particular, for each zone, the comfort constraints are defined as the 95% envelop of the zone temperatures controlled by BC from December 2011 to February 2012. In other words, the chosen thermal comfort lower bounds are violated by the BC controller with probability of 5% at
each time instant. Our simulation focuses on HVAC systems (Figure 1) operating on heating seasons, thus only the lower comfort bounds are of interest, and the upper bounds are always satisfied for well-controlled HVAC systems. We also remark that by enlarging these constraints, any of the five MPC controllers will improve its performance compared to the BC.

Figure 8(a) and Figure 8(b) illustrate the comfort bounds for two of the eight zones in Bancroft library at UC Berkeley. The dots in Figure 8 represents the daily historical zone temperatures, and the solid lines are the computed comfort bounds (5% of the dots are outside the comfort sets at each time step).

![Figure 8](image)

**Fig. 8.** Thermal comfort bounds

The constraints on supply air mass flow rate and supply air temperature are also learned from historical data from December 2011 to February 2012. The upper bounds and lower bounds of supply air mass flow rate are computed as the point-wise max and min of the historical profiles. The lower bounds of airflow rates guarantee minimum ventilation levels during occupied hours. The maximum achievable supply air temperature is limited by the hot water temperature through heating coils and the physical characteristics of coils. It is assumed to be time invariant for each individual VAV box.

2) **Comparison Metrics:** The performance of the model predictive controllers L1–L5 is evaluated by closed-loop simulation with system model (11a) for fifty five days. In the simulation, the disturbance realizations are recorded measurements from December 2011 to February 2012. Three metrics are proposed to compare L1–L5 with the baseline control logic.

- **Closed-loop energy savings compared to baseline control**

  \[
  S^\diamond = \frac{E_{tot}^{BC} - E_{tot}^\diamond}{E_{tot}^{BC}}, \tag{50}
  \]

  where \(S^\diamond\) is the energy saving for controller \(\diamond\), and \(E_{tot}^{BC}\) is the total energy consumption

  \[
  E_{tot} = \sum_{t_s}^{t_f} P_{tot}(x_t, u^*_t, w_t) \Delta t, \tag{51}
  \]

  where \(t_s\) is the simulation start time, \(t_f\) is the simulation end time, \(u^*_t\) is the implemented control input at time \(t\), and \(w_t\) is the disturbance realization at time \(t\) in closed-loop simulation.
Total comfort improvement compared to baseline control

\[ \Delta^\circ = \frac{V_{BC}^{\text{tot}} - V_{\text{tot}}^{\circ}}{V_{BC}^{\text{tot}}}, \]  

(52)

where \( \Delta^\circ \) is the comfort improvement for controller \( \circ \), and \( V_{\text{tot}}^{\circ} \) is the total comfort violation

\[ V_{\text{tot}} = \sum_{t_f}^{t_e} \sum_{i \in \mathcal{V}} \max(T^i(t) - T^i(t), 0) \Delta t, \]  

(53)

Thermal efficiency of the HVAC system \( \eta \)

\[ \eta = \frac{E_{\text{thermal}}^{\text{tot}}}{E_{\text{tot}}}, \]  

(54)

where \( E_{\text{thermal}}^{\text{tot}} \) is the total thermal energy delivered by HVAC system, defined as

\[ E_{\text{thermal}}^{\text{tot}} = \sum_{t_e}^{t_f} (P_c/p_c + P_h/p_h) \Delta t. \]  

(55)

3) Simulation Results: Table I summarizes the results of our tests.

<table>
<thead>
<tr>
<th>Controller</th>
<th>PMPC</th>
<th>CMPC</th>
<th>GSMPc</th>
<th>ESMPC</th>
<th>SSMPC</th>
</tr>
</thead>
<tbody>
<tr>
<td>Energy savings ( S ) (%)</td>
<td>22.54</td>
<td>22.43</td>
<td>20.39</td>
<td>20.33</td>
<td>19.93</td>
</tr>
<tr>
<td>comfort improvement ( \Delta ) (%)</td>
<td>96.25</td>
<td>-121.91</td>
<td>17.26</td>
<td>88.07</td>
<td>93.97</td>
</tr>
</tbody>
</table>

Table I compares the comfort improvement \( \Delta \) and the total energy savings \( S \). The tradeoff between performance, conservatism, and complexity is clear. With perfect knowledge of future predictions, PMPC presents the highest comfort improvement while achieving maximum energy savings of 22.54%. The simplest MPC controller to implement, CMPC, achieves comparable energy savings of 22.43%. However, the zone temperature regulated by CMPC violates the comfort constraints 121.17% more than the baseline control. CMPC ignores the disturbance uncertainty at the design stage and this lead to the incapability of the CMPC to satisfy the comfort constraints.

The performance of the ESMPC and SSMPC proposed in this paper is not too far from the PMPC. This confirms the effectiveness of our approach. We notice that the ESMPC consumes less energy than SSMPC at the cost of more comfort violations. This is due to a more conservative approximation of chance constraints in the SSMPC approach presented in Algorithm 1 compared to discrete convolution algorithm 2.

GSMPc is computationally simpler than ESMPC and SSMPC to compute the chance constraint offset \( \alpha \) in (41). Compared to the SSMPC and ESMPC methods proposed in Section III, GSMPc achieves similar energy savings while violating more comfort constraints. The coarse Gaussian approximations of disturbance is the reason for this. The approximation error of the load probability distribution function is illustrated in Figure 9. In particular, Figure 9(a) depicts the load Gaussian model (dashed lines) compared to a load model which uses finitely-supported density approximation (solid lines) using Gaussian kernels [39]. Figure 9(b) shows the cumulative density function of the PDF in Figure 9(a). The horizontal dash-dotted line in Figure 9(b) indicates the 95% confidence level. With
a 95% confidence level, the cumulative density function of the Gaussian model underestimates the tail. This leads to an underestimation of the tightening offsets $\alpha$ in (33) and thus a higher probability of comfort violations than that specified by chance constraints (13c).

In the ESMPC and SSMPC controllers the energy savings compared to baseline control can be explained as following two mechanisms. Firstly, the zone temperatures controlled by ESMPC and SSMPC are closer to the lower bounds (this can be observed in Figure 10). The dots in Figure 10(a) and 10(b) are the zone temperatures controlled by BC, the cross markers in Figure 10(a) and 10(b) represent the zone temperatures controlled by ESMPC and SSMPC, respectively. Secondly, the ESMPC and SSMPC controllers also optimize the combination of supply airflow rate and supply air temperature so that the required heating energy is delivered with minimum energy consumption. Table II listed the thermal efficiency of the HVAC system controlled by L1–L5. It can be noticed that the coordination aspects of the energy-savings in MPC is minimal, and the BC controller has similar performance. This implies that the reason for SMPC consuming less energy than BC is mainly because SMPC controls the zone temperature closer to the lower bounds.

The results in Table I show the advantages of stochastic MPC formulation compared to nominal MPC with expected forecasts. Next we show that this depends on the uncertainty level. We compare the energy consumption and comfort violations of CMPC and SSMPC with increasing levels of load uncertainty. At the design stage,
CMPC ignores the uncertainty of the load predictions, and only considers the mean value of predictions. On the other hand, SSMPC seeks control signals to guarantee the level of comfort satisfaction while respecting the uncertain load predictions. In this section, the load uncertainty level is scaled by the parameter $\vartheta$ as follows. For SSMPC, we scale the independent disturbance samples $\{\tilde{w}_k^1, \tilde{w}_k^2, ..., \tilde{w}_k^{N_s,k}\}$ extracted from the probability density function of the load uncertainty as $\{\vartheta \tilde{w}_k^1, \vartheta \tilde{w}_k^2, ..., \vartheta \tilde{w}_k^{N_s,k}\}$. The scaled set of samples is used to compute the tightening offsets in Algorithm 2. For CMPC, the resulting control policy is independent of the scaling of load uncertainty as CMPC only takes into account the mean value of the load prediction at the design stage. The load realizations in closed-loop simulation have to be scaled accordingly,

$$w(t)_a = \tilde{w}(t) + \vartheta (w(t) - \hat{w}(t))$$

where $w(t)_a$ is the scaled load realization, $w(t)$ is the original load realization, and $\hat{w}(t)$ is the weekly mean load computed as Figure 5 in Section II-C.

An increasing value of the scaling factor $\vartheta$ indicates the increase of disturbance prediction errors. For example, weather forecast information from an inferior weather station results in a high value of $\vartheta$.

Figure 11 reports the simulation results of CMPC and SSMPC for eight zones in the second floor of the Bancroft library from December 1, 2011 to February 1, 2012. Figure 11(a) shows the energy consumption ration $E_{\text{CMPC}}^{\text{tot}} / E_{\text{SSMPC}}^{\text{tot}}$ with increasing level of the load uncertainty, and Figure 11(b) shows the comfort violation ration $V_{\text{CMPC}}^{\text{tot}} / V_{\text{SSMPC}}^{\text{tot}}$ with increasing level of load uncertainty. It is observed that at low levels of load uncertainty ($\vartheta < 5$), SSMPC is able to improve the comfort satisfaction compared to CMPC while consuming comparable energy. However, for load uncertainty scaling factors $\vartheta > 5$, SSMPC loses its advantage, and CMPC shows comparable performance to SSMPC despite its simplicity at design stage.

**B. Complexity Analysis for SSMPC and ESMPC**

This section focuses on the complexity of SSMPC and ESMPC when transforming the chance constraints (13c) into the corresponding deterministic ones (32). Computational demands of Algorithms 1 and 2 are analyzed with respect to three problem size parameters: number of thermal zones $N_v$, prediction horizon length $T$ and number of samples $N_s$.

The discrete convolutions in Step 3 are the most computationally demanding operations in Algorithm 1. The complexity of discrete convolution is $O(N_s^2)$, where $N_s$ is the number of samples used to discretize the probability density functions in (40). The number of discrete convolutions needed to approximate the probability density function $pdf_{\tilde{w}_k}$ is a linear function of $N_v T$, where $N_v$ is the number of zones and $T$ is the number of prediction steps.
Algorithm 1 is repeated for $N_c = 2N_vT$ number of chance constraints in (13c), thus the complexity of the discrete method to compute the offsets in (13) is of the order $O(N_s^2N_v^2T^2)$.

If we assume that the $N_v$ thermal zones are, in the worst case, fully coupled, Algorithm 2 has complexity $O(N_v^2)$ (see Step S2.2 of Algorithm 2). Similarly, the algorithm complexity as a function of the prediction horizon $T$ is $O(T^2)$. With respect to the number of samples $N_s$, the most computationally demanding step of Algorithm 2 is sorting the offsets $\alpha_{j,k}^{\ell,\tau}$ in (45). This has complexity of $O(N_s \log N_s)$, if a quick-sort algorithm is used. Taking all above mentioned into account we can conclude that overall complexity of Algorithm 2 is $O(N_s \log N_s N_v^2T^2)$ while the complexity of Algorithm 1 is $O(N_s^2N_v^2T^2)$.

Figures 12 shows the execution times associated with the formulation of ESMPC and SSMPC as functions of the prediction horizon length $T$. Similar results can be obtained if we increase the number of zones $N_v$. In Figure 12, the solid line represents the average computational time for discrete method (Algorithm 1), and the dashed line represents the average computational time for sample-based method (Algorithm 2). While the ESMPC behaves as expected, the SSMPC shows a super-linear relationship between the SSMPC execution time and the prediction horizon length. The reason for such the discrepancy between theoretical and simulation results is thought to be linked to the sparsity of the matrices involved in calculation of the constraint offsets in Step S2.2 of Algorithm 2. On the other hand, a discrete convolution operation in Step 3 of Algorithm 1 involves a dense matrix-vector multiplication and thus, the complexity of this operation is $O(T^2)$, regardless of the potential system sparsity. We remark that ESMPC and SSMPC solve the same optimization problem (49) with the exception of the tightening offset $\alpha_{j,k}^{\ell,\tau}$. For the size and complexity of problems considered in our study, the average time to solve Problem (49) with Ipopt is five seconds.

C. Experimental Results

The proposed ESMPC controller has been implemented to control the MPC lab at UC Berkeley. The MPC lab resides on the underground floor of Etcheverry building, it has no windows, and thus is subject to negligible solar
Fig. 12. Computational time for ESMPC and SSMPC

radiation. The dimensions of the lab are $9 \text{m} \times 9 \text{m} \times 3.5 \text{m}$, and it hosts fourteen students with fourteen desktop PCs running during weekdays. The VAV box serving the lab is depicted in Figure 13. The inlet air is the cool or warm air supplied by the central air handling unit in the Etcheverry building, and we do not control its temperature. The VAV box in the lab consists of an inlet air damper to control the inlet air flow rate, a set of cooling/heating coils to cool down or warm up the supply air, a supply fan to maintain the static pressure in the duct, a supply air damper to regulate the supply air flow rate, and a return air damper to balance the air pressure in the lab.

Fig. 13. Scheme plot for HVAC system in the lab

The lab is equipped with a modern digital control system which allows us to monitor and analyze the performance. In the following sections, we present the system identification results for the lab, the implementation details for the ESMPC controller, and some experimental results.

1) System Identification: The thermal dynamics of the lab are modeled by the bilinear regression model (3) with $q_x = q_d = 2$. The regression parameters are fitted using historical data from weekend hours on September 27 and
28, 2012, since there is negligible occupancy load during weekends in the lab. Figure 15 depicts the identification results, where the dash dot line is the predicted lab temperature and the solid line represents the measured lab temperature. The identified model parameters are reported in Table III. The load uncertainty model is computed as explained in Section II-C.

Figure 16 shows the estimated weekly disturbance loads from April 1 to July 21, 2012. In Figure 16, the dots are the realizations of the loads while the solid lines is the mean value of the weekly disturbance load. The samples in Figure 16 are used to learn the statistics of the load model.
We use the ambient temperature uncertainty prediction model (6) where the probability density function is depicted in Figure 7 and the mean value of the ambient temperature prediction is downloaded from services provided by National Oceanic and Atmospheric Administration (NOAA).

2) ESMPC Implementation: We use a sampling of 5 minutes, and a prediction horizon of 5 hours. At time $t$, the ESMPC is implemented as follows

1) Obtain the lab temperature sensor readings.
2) Downloads the weather predictions from the weather forecast services (NOAA).
3) Obtain the uncertain load predictions $P_{d,k}$. The uncertain load prediction is modeled as a look-up table learned from samples in Figure ??.
4) Solve the stochastic MPC problem (49) to obtain the optimal setpoint for the supply air mass flow rate and supply air temperature. The optimization problem is solved using Ipopt on a single Intel Core Duo CPU 3.00GHz.
5) Send the optimal setpoint to low-level PID controllers which regulate the coil valve positions and the fan speed to track the set points.

3) Experimental Results: Figure 17 reports the lab temperature controlled by ESMPC (solid line) from January 23 to January 30, 2013. In Figure 17 the dashed lines are the lower and upper bounds of the comfort region. The lower (upper) bound of the comfort set is 21.1 (23.9) °C for unoccupied hours from 16:00 to 06:00, and 22.3 (23.3) °C for occupied hours from 06:00 to 16:00 everyday. It is observed that ESMPC maintains the zone temperature (solid line) within the comfort region, and the thermal comfort is guaranteed despite the uncertain load predictions in Figure 16.

In Figure 18, the solid line is the supply air temperature delivered to the lab, and the dash-dot line is the inlet air temperature delivered by the central AHU in Etcheverry building. It is observed that the supply air temperature is kept higher than the inlet air temperature, which is required for heating seasons.

The supply air mass flow rate is depicted in Figure 19. The lab requires a minimum of 600 cfm supply air flow rate during the unoccupied hours, and 800 cfm for occupied hours to meet the minimum ventilation requirement.

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Fig. 16. Lab thermal load model
V. Conclusions

We presented a stochastic model predictive control (SMPC) design methodology for HVAC systems. The SMPC uses uncertain prediction of weather conditions and building loads to minimize the expected energy consumption, satisfy the robust operational constraints, and provide guarantees on the probability of comfort violations. We have shown how to model the building thermal zones as a network of bilinear systems. The uncertainty models for occupancy loads and weather predictions were modeled as finitely-supported probability distribution functions learned from historical data. In order to reduce the conservatism of the stochastic MPC scheme while retaining computational tractability, we proposed a feedback linearization scheme. The chance constraints are then transformed to deterministic ones using two techniques: discrete convolution integrals and a sample-based method.

Simulation and experiments have shown the effectiveness of the proposed approach. In particular, we have highlighted the tradeoff between performance, conservatism, and complexity of model predictive control algorithms for HVAC systems.
Fig. 19. Lab supply air mass flow rate

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REFERENCES


