Model Predictive Control
Reachability and Invariance

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**Definitions (Polyhedra and polytopes)**

A **polyhedron** is the intersection of a *finite* number of closed halfspaces:

\[ Z = \{ z \mid a_1^\top z \leq b_1, a_2^\top z \leq b_2, \ldots, a_m^\top z \leq b_m \} = \{ z \mid Az \leq b \} \]

where \( A := [a_1, a_2, \ldots, a_m]^\top \) and \( b := [b_1, b_2, \ldots, b_m]^\top \).

A **polytope** is a *bounded* polyhedron.

Polyhedra and polytopes are always convex.

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**General Set Definitions and Operations**

- An *n*-dimensional ball \( B(x_0, \rho) \) is the set \( B(x_0, \rho) = \{ x \in \mathbb{R}^n \mid \sqrt{\|x - x_0\|^2} \leq \rho \} \). \( x_0 \) and \( \rho \) are the center and the radius of the ball, respectively.

- The convex combination of \( x_1, \ldots, x_k \) is defined as the point \( \lambda_1 x_1 + \ldots + \lambda_k x_k \) where \( \sum_{i=1}^k \lambda_i = 1 \) and \( \lambda_i \geq 0, \ i = 1, \ldots, k \).

- The convex hull of a set \( K \subseteq \mathbb{R}^n \) is the set of all convex combinations of points in \( K \) and it is denoted as \( \text{conv}(K) \):

\[
\text{conv}(K) \triangleq \{ \lambda_1 x_1 + \ldots + \lambda_k x_k \mid x_i \in K, \ \lambda_i \geq 0, \ i = 1, \ldots, k, \ 
\sum_{i=1}^k \lambda_i = 1 \}.
\]
1. Polyhedra and Polytopes

1.1 General Set Definitions and Operations

Polyhedra Representations

An $\mathcal{H}$-polyhedron $\mathcal{P}$ in $\mathbb{R}^n$ denotes an intersection of a finite set of closed halfspaces in $\mathbb{R}^n$:

$$\mathcal{P} = \{ x \in \mathbb{R}^n : Ax \leq b \}$$

In Matlab: $\mathcal{P} = \text{Polytope}(A,b)$

A two-dimensional $\mathcal{H}$-polyhedron

Inequalities which can be removed without changing the polyhedron are called redundant. The representation of an $\mathcal{H}$-polyhedron is minimal if it does not contain redundant inequalities.

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Polyhedra Representations

- A $\mathcal{V}$-polytope $\mathcal{P}$ in $\mathbb{R}^n$ is defined as
  $$\mathcal{P} = \text{conv}(V)$$
  for some $V = [V_1, \ldots, V_k] \in \mathbb{R}^{n \times k}$.
- Any $\mathcal{H}$-polytope is a $\mathcal{V}$-polytope and viceversa.
- A polytope $\mathcal{P} \subset \mathbb{R}^n$, is full-dimensional if it is possible to fit a non-empty $n$-dimensional ball in $\mathcal{P}$
- If $\|A_i\|_2 = 1$, where $A_i$ denotes the $i$-th row of a matrix $A$, we say that the polytope $\mathcal{P}$ is normalized.
Polyhedra Representations

- The faces of dimension 0 and 1 are called vertices and edges, respectively.

![V-representation](image1.png)  ![H-representation](image2.png)

(a) V-representation.  (b) H-representation.

Polytopal Complexes

A set $\mathcal{C} \subseteq \mathbb{R}^n$ is called a P-collection (in $\mathbb{R}^n$) if it is a collection of a finite number of $n$-dimensional polytopes, i.e.

$$\mathcal{C} = \{C_i\}_{i=1}^{N_C},$$

where $C_i := \{x \in \mathbb{R}^n : C_i^x x \leq C_i^c\}$, $\dim(C_i) = n$, $i = 1, \ldots, N_C$, with $N_C < \infty$.

In Matlab: $Q = [P1, P2, P3]$, $R = [P4, Q, [P5, P6], P7]$
Functions on Polytopal Complexes

- A function $h(\theta) : \Theta \rightarrow \mathbb{R}^k$, where $\Theta \subseteq \mathbb{R}^s$, is piecewise affine (PWA) if there exists a strict partition $R_1, \ldots, R_N$ of $\Theta$ and $h(\theta) = H^i \theta + k^i$, $\forall \theta \in R_i$, $i = 1, \ldots, N$.

- A function $h(\theta) : \Theta \rightarrow \mathbb{R}^k$, where $\Theta \subseteq \mathbb{R}^s$, is piecewise affine on polyhedra (PPWA) if there exists a strict polyhedral partition $R_1, \ldots, R_N$ of $\Theta$ and $h(\theta) = H^i \theta + k^i$, $\forall \theta \in R_i$, $i = 1, \ldots, N$.

- A function $h(\theta) : \Theta \rightarrow \mathbb{R}$, where $\Theta \subseteq \mathbb{R}^s$, is piecewise quadratic (PWQ) if there exists a strict partition $R_1, \ldots, R_N$ of $\Theta$ and $h(\theta) = \theta' H^i \theta + k^i \theta + l^i$, $\forall \theta \in R_i$, $i = 1, \ldots, N$.

- A function $h(\theta) : \Theta \rightarrow \mathbb{R}$, where $\Theta \subseteq \mathbb{R}^s$, is piecewise quadratic on polyhedra (PPWQ) if there exists a strict polyhedral partition $R_1, \ldots, R_N$ of $\Theta$ and $h(\theta) = \theta' H^i \theta + k^i \theta + l^i$, $\forall \theta \in R_i$, $i = 1, \ldots, N$. 

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Basic Operations on Polytopes

- Convex Hull of a set of points \( V = \{ V_i \}_{i=1}^{N_V} \), with \( V_i \in \mathbb{R}^n \),

\[
\text{conv}(V) = \{ x \in \mathbb{R}^n : x = \sum_{i=1}^{N_V} \alpha_i V_i, \ 0 \leq \alpha_i \leq 1, \ \sum_{i=1}^{N_V} \alpha_i = 1 \}. \quad (1)
\]

In Matlab: \( P = \text{hull}(V) \), \( V \) matrix containing vertices of the polytope \( P \)

- Vertex Enumeration of a polytope \( P \) given in \( H \)-representation. (dual of the convex hull operation)

  In Matlab: \( V = \text{extreme}(P) \)

  Used to switch from a \( V \)-representation of a polytope to an \( H \)-representation.

- Polytope reduction is the computation of the minimal representation of a polytope. A polytope \( P \subset \mathbb{R}^n \), \( P = \{ x \in \mathbb{R}^n : Ax \leq b \} \) is in a minimal representation if the removal of any row in \( Ax \leq b \) would change it (i.e., if there are no redundant constraints).

  In Matlab: \( P = \text{Polytope}(A,b,\text{normal},\text{minrep}) \), \( \text{minrep}=1 \)

- The Chebychev Ball of a polytope \( P \) corresponds to the largest radius ball \( B(x_c, R) \) with center \( x_c \), such that \( B(x_c, R) \subset P \).

  In Matlab: \( P.xCheb, P.rCheb \)
Basic Operations on Polytopes

- **Projection**
  Given a polytope \( \mathcal{P} = \{ [x^T y^T]^T \in \mathbb{R}^{n+m} : A^T x + A^y y \leq b \} \subset \mathbb{R}^{n+m} \) the projection onto the \( x \)-space \( \mathbb{R}^n \) is defined as

\[
\text{proj}_x(\mathcal{P}) := \{ x \in \mathbb{R}^n | \exists y \in \mathbb{R}^m : A^T x + A^y y \leq b \}.
\]

In Matlab: \( Q = \text{projection}(P, \text{dim}) \)

---

Affine Mappings and Polyhedra

- Consider a polyhedron \( \mathcal{P} = \{ x \in \mathbb{R}^n | Hx \leq k \} \), with \( H \in \mathbb{R}^{np \times n} \) and an affine mapping \( f(z) \)

\[
f : z \in \mathbb{R}^n \mapsto Az + b, \quad A \in \mathbb{R}^{n \times n}, \quad b \in \mathbb{R}^n
\]

- Define the composition of \( \mathcal{P} \) and \( f \) as the following polyhedron

\[
\mathcal{P} \circ f \triangleq \{ z \in \mathbb{R}^n | Hf(z) \leq k \} = \{ z \in \mathbb{R}^m | HAz \leq k - Hb \}
\]

- Useful for backward-reachability
Affine Mappings and Polyhedra

Consider a polyhedron $\mathcal{P} = \{x \in \mathbb{R}^n \mid Hx \leq k\}$, with $H \in \mathbb{R}^{np \times n}$ and an affine mapping $f(z)$

$$f : z \in \mathbb{R}^n \mapsto Az + b, \ A \in \mathbb{R}^{n \times n}, \ b \in \mathbb{R}^n$$

Define the composition of $f$ and $\mathcal{P}$ as the following polyhedron

$$f \circ \mathcal{P} \triangleq \{ y \in \mathbb{R}^n \mid y = Ax + b \ \forall x \in \mathbb{R}^n, \ Hx \leq k \}$$

The polyhedron $f \circ \mathcal{P}$ in can be computed as follows. Write $\mathcal{P}$ in $V$-representation $\mathcal{P} = \text{conv}(V)$ and map the vertices $V = \{V_1, \ldots, V_k\}$ through the transformation $f$. Because the transformation is affine, the set $f \circ \mathcal{P}$ is the convex hull of the transformed vertices

$$f \circ \mathcal{P} = \text{conv}(F), \ F = \{AV_1 + b, \ldots, AV_k + b\}.$$ 

Useful for forward-reachability
Set Definition

We consider the following two types of systems autonomous systems:

$$x(t+1) = f_a(x(t)),$$

(2)

and systems subject to external inputs:

$$x(t+1) = f(x(t), u(t)).$$

(3)

Both systems are subject to state and input constraints

$$x(t) \in \mathcal{X}, \ u(t) \in \mathcal{U}, \ \forall \ t \geq 0.$$ 

The sets $\mathcal{X}$ and $\mathcal{U}$ are polyhedra and contain the origin in their interior.
Reach Set Definition

For the autonomous system (2) we denote the one-step reachable set as

\[ \text{Reach}(S) \triangleq \{ x \in \mathbb{R}^n : \exists x(0) \in S \text{ s.t. } x = f_a(x(0)) \} \]

For the system (3) with inputs we denote the one-step reachable set as

\[ \text{Reach}(S) \triangleq \{ x \in \mathbb{R}^n : \exists x(0) \in S, \exists u(0) \in U \text{ s.t. } x = f(x(0), u(0)) \} \]

Pre Set Definition

“Pre” sets are the dual of one-step reachable sets. The set

\[ \text{Pre}(S) \triangleq \{ x \in \mathbb{R}^n : f_a(x) \in S \} \]

defines the set of states which evolve into the target set \( S \) in one time step for the system (2).

Similarly, for the system (3) the set of states which can be driven into the target set \( S \) in one time step is defined as

\[ \text{Pre}(S) \triangleq \{ x \in \mathbb{R}^n : \exists u \in U \text{ s.t. } f(x, u) \in S \} \]
Pre Set Computation - Autonomous Systems

Assume the system is linear and autonomous

\[ x(t + 1) = Ax(t) \]

Let

\[ S = \{ x : Hx \leq h \}, \]

Then the set \( \text{Pre}(S) \) is

\[ \text{Pre}(S) = \{ x : HAx \leq h \} \]

Note that by using polyhedral notation, the set \( \text{Pre}(S) \) is simply \( S \circ A \).
Reach Set Computation - Autonomous Systems

The set $\text{Reach}(S)$ is obtained by applying the map $A$ to the set $S$. Write $S$ in $V$-representation

$$S = \text{conv}(V) \quad (5)$$

and map the set of vertices $V$ through the transformation $A$. Because the transformation is linear, the reach set is simply the convex hull of the transformed vertices

$$\text{Reach}(S) = A \circ S = \text{conv}(AV) \quad (6)$$

Pre Set Computation - System with Inputs

Consider the system

$$x(t+1) = Ax(t) + Bu(t)$$

Let

$$S = \{x \mid Hx \leq h\}, \quad U = \{u \mid H_u u \leq h_u\}, \quad (7)$$

The Pre set is

$$\text{Pre}(S) = \left\{ x \in \mathbb{R}^n \mid \exists u \in \mathbb{R} \text{ s.t.} \begin{bmatrix} HA & HB \\ 0 & H_u \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \leq \begin{bmatrix} h \\ h_u \end{bmatrix} \right\}$$

which is the projection onto the $x$-space (with dimension $\mathbb{R}^n$) of the polyhedron

$$\mathcal{T} := \left\{ \begin{bmatrix} HA & HB \\ 0 & H_u \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \leq \begin{bmatrix} h \\ h_u \end{bmatrix} \right\}.$$ 

In Matlab: $Q = \text{projection}(\mathcal{T}, n)$
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Controllable Sets

Definition ($N$-Step Controllable Set $\mathcal{K}_N(\mathcal{O})$)

For a given target set $\mathcal{O} \subseteq \mathcal{X}$, the $N$-step controllable set $\mathcal{K}_N(\mathcal{O})$ is defined as:

$$\mathcal{K}_N(\mathcal{O}) \triangleq \text{Pre}(\mathcal{K}_{N-1}(\mathcal{O})) \cap \mathcal{X}, \quad \mathcal{K}_0(\mathcal{O}) = \mathcal{O}, \quad N \in \mathbb{N}^+.$$

All states $x_0 \in \mathcal{K}_N(\mathcal{O})$ can be driven, through a time-varying control law, to the target set $\mathcal{O}$ in $N$ steps, while satisfying input and state constraints.

Definition (Maximal Controllable Set $\mathcal{K}_\infty(\mathcal{O})$)

For a given target set $\mathcal{O} \subseteq \mathcal{X}$, the maximal controllable set $\mathcal{K}_\infty(\mathcal{O})$ for the system $x(t+1) = f(x(t), u(t))$ subject to the constraints $x(t) \in \mathcal{X}$, $u(t) \in \mathcal{U}$ is the union of all $N$-step controllable sets contained in $\mathcal{X}$ ($N \in \mathbb{N}$).
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2.4 $N$-Step Reachable Sets

$N$-Step Reachable Sets

**Definition ($N$-Step Reachable Set $\mathcal{R}_N(\mathcal{X}_0)$)**

For a given initial set $\mathcal{X}_0 \subseteq \mathcal{X}$, the $N$-step reachable set $\mathcal{R}_N(\mathcal{X}_0)$ is

$$\mathcal{R}_{i+1}(\mathcal{X}_0) \triangleq \text{Reach} (\mathcal{R}_i(\mathcal{X}_0)), \quad \mathcal{R}_0(\mathcal{X}_0) = \mathcal{X}_0, \quad i = 0, \ldots, N - 1$$

All states $x_0 \in \mathcal{X}_0$ can will evolve to the $N$-step reachable set $\mathcal{R}_N(\mathcal{X}_0)$ in $N$ steps

Same definition of Maximal Reachable Set $\mathcal{R}_\infty(\mathcal{X}_0)$ can be introduced.
Outline

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Invariant Sets

Invariant sets

- are computed for autonomous systems
- for a given feedback controller \( u = g(x) \), provide the set of initial states whose trajectory will never violate the system constraints.

**Definition (Positive Invariant Set)**

A set \( O \subseteq X \) is said to be a positive invariant set for the autonomous system \( x(t + 1) = f_a(x(t)) \) subject to the constraints \( x(t) \in \mathcal{X} \), if

\[
x(0) \in O \implies x(t) \in O, \quad \forall t \in \mathbb{N}^+
\]

**Definition (Maximal Positive Invariant Set \( O_\infty \))**

The set \( O_\infty \) is the maximal invariant set if \( O_\infty \) is invariant and \( O_\infty \) contains all the invariant sets contained in \( \mathcal{X} \).

### Algorithm

**Input:** \( f_a, \mathcal{X} \)  
**Output:** \( O_\infty \)

1. let \( \Omega_0 = \mathcal{X} \),
2. let \( \Omega_{k+1} = \text{Pre}(\Omega_k) \cap \Omega_k \)
3. if \( \Omega_{k+1} = \Omega_k \) then \( O_\infty \leftarrow \Omega_{k+1} \)
4. else go to 2

The algorithm generates the set sequence \( \{ \Omega_k \} \) satisfying \( \Omega_{k+1} \subseteq \Omega_k \), \( \forall k \in \mathbb{N} \) and it terminates when \( \Omega_{k+1} = \Omega_k \) so that \( \Omega_k \) is the maximal positive invariant set \( O_\infty \) for \( x(t + 1) = f_a(x(t)) \).
Control Invariant Sets

**Control** invariant sets

- are computed for systems *subject to external inputs*
- provide the set of initial states for which *there exists* a controller such that the system constraints are never violated.

**Definition (Control Invariant Set)**

A set $\mathcal{C} \subseteq \mathcal{X}$ is said to be a control invariant set if

$$x(t) \in \mathcal{C} \implies \exists u(t) \in \mathcal{U} \text{ such that } f(x(t), u(t)) \in \mathcal{C}, \quad \forall t \in \mathbb{N}^+$$

**Definition (Maximal Control Invariant Set $\mathcal{C}_\infty$)**

The set $\mathcal{C}_\infty$ is said to be the maximal control invariant set for the system $x(t + 1) = f(x(t), u(t))$ subject to the constraints in $x(t) \in \mathcal{X}, \ u(t) \in \mathcal{U}$, if it is control invariant and contains all control invariant sets contained in $\mathcal{X}$. 
### Control Invariant Sets

Same geometric condition for control invariants holds: $C$ is a control invariant set if and only if

$$C \subseteq \text{Pre}(C)$$  \hspace{1cm} (8)

#### Algorithm

<table>
<thead>
<tr>
<th>Step</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>let $\Omega_0 = \mathcal{X}$,</td>
</tr>
<tr>
<td>2</td>
<td>let $\Omega_{k+1} = \text{Pre}(\Omega_k) \cap \Omega_k$</td>
</tr>
<tr>
<td>3</td>
<td>if $\Omega_{k+1} = \Omega_k$ then $C_\infty \leftarrow \Omega_{k+1}$</td>
</tr>
<tr>
<td>4</td>
<td>else go to 2</td>
</tr>
</tbody>
</table>

The algorithm generates the set sequence $\{\Omega_k\}$ satisfying $\Omega_{k+1} \subseteq \Omega_k$, $\forall k \in \mathbb{N}$ and it terminates if $\Omega_{k+1} = \Omega_k$ so that $\Omega_k$ is the maximal control invariant set $C_\infty$ for the constrained system.

### Invariant Sets and Control Invariant Sets

- The set $\mathcal{O}_\infty$ ($C_\infty$) is **finitely determined** if and only if $\exists i \in \mathbb{N}$ such that $\Omega_{i+1} = \Omega_i$.
- The smallest element $i \in \mathbb{N}$ such that $\Omega_{i+1} = \Omega_i$ is called the **determinedness index**.
- For all states contained in the maximal control invariant set $C_\infty$ there exists a control law, such that the system constraints are never violated.